Coupled nonlinear Schrödinger field equations for electromagnetic wave propagation in nonlinear left-handed materials

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For an isotropic and homogeneous nonlinear left-handed materials, for which the effective medium approximation is valid, Maxwell's equations for electric and magnetic fields lead naturally, within the slowly varying envelope approximation, to a system of coupled nonlinear Schrödinger equations. This system is equivalent to the well-known Manakov model that under certain conditions, is completely integrable and admits bright and dark soliton solutions. It is demonstrated that left- and right-handed (normal) nonlinear mediums may have compound bright and dark soliton solutions, respectively. These results are also supported by numerical calculations.

Recently the study of the electromagnetic (EM) properties of artificial complex media with simultaneously negative dielectric permittivity ϵ_{eff} and magnetic permeability μ_{eff} has been the subject of great attention. Such media are usually referred to as left-handed materials (LHMs),[1] and they demonstrate a number of peculiar properties: reversal of Snell's law of refraction, reversal of the Doppler shift, counter-directed Cherenkov radiation cone, negative refraction index, the refocusing of EM waves from a point sourse, etc. The above-mentioned properties follow directly from Maxwell's equations with appropriate constitutive relations. Pendry[2] has proposed the intriguing possibility to exploit the negative refraction index property of the LHMs in order to overcome known problems with common lenses to achieve a perfect lens that would focus both the propagating as well as the evanescent spectra.

Typical LHM are composed of a combination of a regular array of electrically small resonant particles referred to as split-ring resonators (SRR's) and a regular array of conducting wires,[2–6] responsible for the negative μ_{eff} and ϵ_{eff}, respectively. The size and spacing of the conducting elements of which the medium is composed is assumed to be on a scale much smaller than the wavelengths in the frequency range of interest, so that the composite medium may be considered as a continuous and a homogeneous one (effective medium approximation). Thus far, almost all properties of LHMs were studied in the linear regime of wave propagation, when both ϵ_{eff} and μ_{eff} are considered to be independent of the field intensities. However, nonlinear effects in LHMs have been recently taken under consideration by some authors.[7–9] Zharov *et al*,[7] considered a two-dimensional periodic structure created by arrays of wires and SRR's embedded into a nonlinear dielectric, and they calculated ϵ_{eff} and μ_{eff} for a Kerr-type dielectric permitivity. They showed that the magnetic field intensity couples to the magnetic resonance of the SRR in a nontrivial way, and that changing the material properties from left- to right-handed and back is allowed by varying the field intensity. The study of the nonlinear properties of LHMs could facilitate future efforts in creating tunable structures where the field intensity changes their transmission properties.

In the present work we show that for an isotropic, homogeneous, quasi-one-dimensional LHM, Maxwell's equations with nonlinear constitutive relations lead naturally to a system of coupled nonlinear Schrödinger (CNLS) equations for the envelopes of the propagating electric and magnetic fields. This system is equivalent to the Manakov model[10] that under certain conditions, admits soliton solutions consisting of two components (vector solitons). For specific parameter choices, corresponding to either a left-handed or a right-handed medium, we find compound dark and bright soliton solutions, respectively. The constitutive relations can be generally written as

$$\mathbf{D} = \epsilon_{eff}\mathbf{E} = \epsilon\mathbf{E} + \mathbf{P_{NL}} \tag{1}$$

$$\mathbf{B} = \mu_{eff}\mathbf{H} = \mu\mathbf{H} + \mathbf{M_{NL}}, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities, respectively, $\mathbf{D}$ is the electric flux density, and $\mathbf{B}$ is the magnetic induction. The linear dielectric and magnetic responses of the LHM are described by ϵ and μ, respectively, while $\mathbf{P}_{NL} = \epsilon_{NL}\mathbf{E}$ and $\mathbf{M}_{NL} = \mu_{NL}\mathbf{H}$ are the nonlinear electric polarization and the nonlinear magnetization of the medium, respectively.

It is known that ϵ_{eff} and μ_{eff} in a LHM have to be dispersive, otherwise the energy density could be negative.[1] Their frequency dispersion, including nonlinear effects (but neglecting losses), is given by[7]

$$\epsilon_{eff}(\omega) = \epsilon_0\left(\epsilon_D(|E|^2) - \frac{\omega_p^2}{\omega^2}\right) \tag{3}$$

$$\mu_{eff}(\omega) = \mu_0\left(1 - \frac{F\omega^2}{\omega^2 - \omega_{0NL}^2(|H|^2)}\right), \tag{4}$$

where ω_p is the plasma frequency, F is the filling factor, $\omega_{0NL} = \omega_{0NL}(|H|^2)$ is the nonlinear resonant SRR frequency, and $\epsilon_D(|E|^2) = \epsilon_{D0} + \alpha|E|^2$, with α the strength of the nonlinearity. Positive (negative) α corresponds to a focusing (defocusing) dielectric. For a linear dielectric $\omega_{0NL}(|H|^2) \rightarrow \omega_0$, where ω_0 is the linear resonant SRR frequency. Then Eqs. (3-4) reduce to previously known

expressions.[5,11] The parameters F, ω_p, and ω_0 are related to geometrical and material parameters of the LHM components. Although ϵ_{eff} can be readily put in the form $\epsilon + \epsilon_{NL}(|E|^2)$, for μ_{eff} this is not an obvious task, since $\mu_{eff} = \mu_{eff}(\omega_{0NL})$, and ω_{0NL} depends on $|H|^2$ as[7,8]

$$\alpha\, \Omega^2\, X^6\, |\mathbf{H}|^2 = A^2\, E_c^2\, (1 - X^2)(X^2 - \Omega^2)^2, \qquad (5)$$

where $X = \omega_{0NL}/\omega_0$, $\Omega = \omega/\omega_0$, E_c is a characteristic (large) electric field, and A is a function of physical and geometrical parameters.[7,8] The + (-) sign corresponds to a focusing (defocusing) dielectric. Our α is related to the parameters of Eq. (5) as $\alpha = \pm 1/E_c^2$. Choosing $f_p = \omega_p/2\pi = 10$ GHz and $f_0 = \omega_0/2\pi = 1.45$ GHz, left-handed behavior appears in a narrow frequency band (from $f = 1.45$ GHz to 1.87 GHz). For relatively small fields when μ_{eff} is trully field dependent, one may consider that the magnetic nonlinearity $\mu_{eff} = \mu + \mu_{NL}(|H|^2)$ is of the Kerr type, i.e. $\mu_{NL}(|H|^2) = \beta\, |H|^2$ (Fig. 1). The strength of the magnetic nonlinearity β can be treated as a fitting parameter.

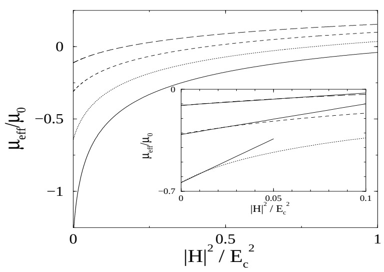

FIG. 1. μ_{eff} as a function of $|H|^2/E_c^2$, for $\Omega = 1.1$ (solid), 1.15 (dotted), 1.20 (dashed), 1.25 (long-dashed), $A = 3$, $F = 0.4$, and $\hat{\alpha} > 0$. Inset: Fitting to a line of the three first curves of the main figure for relatively small fields.

Using Eqs. (1)-(2) and known identities, we get quite general vector wave equations for the fields $\mathbf{E}$ and $\mathbf{H}$

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu\frac{\partial^2\mathbf{P}_{NL}}{\partial t^2} - \nabla(\nabla\cdot\mathbf{E}) =$$
$$\frac{\partial}{\partial t}[(\nabla\mu_{NL})\times\mathbf{H}] + \frac{\partial}{\partial t}\left[\mu_{NL}\frac{\partial}{\partial t}(\epsilon\mathbf{E} + \mathbf{P}_{NL})\right] \qquad (6)$$
$$\nabla^2\mathbf{H} - \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} - \epsilon\frac{\partial^2\mathbf{M_{NL}}}{\partial t^2} - \nabla(\nabla\cdot\mathbf{H}) =$$
$$-\frac{\partial}{\partial t}[(\nabla\epsilon_{NL})\times\mathbf{E}] + \frac{\partial}{\partial t}\left[\epsilon_{NL}\frac{\partial}{\partial t}(\mu\mathbf{H} + \mathbf{M}_{NL})\right]. \qquad (7)$$

Consider an $x-$polarized plane wave with frequency ω propagating along the $z-$ axis:[12]

$$\mathbf{E} = (E(z,t), 0, 0), \qquad \mathbf{H} = (0, H(z,t), 0), \qquad (8)$$

where

$$E(z,t) = q(z,t)e^{i(kz-\omega t)}\ H(z,t) = p(z,t)e^{i(kz-\omega t)}, \qquad (9)$$

with k being the wavenumber. The envelopes $q(z,t)$ and $p(z,t)$ of E and H, respectively, change slowly in z and t. We therefore introduce the slow variables

$$\xi = \varepsilon(z - \omega' t) \qquad \tau = \varepsilon^2 t, \qquad (10)$$

where ε is a small parameter, and $\omega' = \partial\omega/\partial k$ is the group velocity of the wave. Taking into account Eqs. (8-9), substituting slow variables into Eqs. (6-7), assuming that $\alpha = \hat{\alpha}\varepsilon^2$, $\beta = \hat{\beta}\varepsilon^2$, and expressing p and q as an asymptotic expansion in terms of ε[12]

$$q(\xi,\tau) = q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau) + \varepsilon^2 q_2(\xi,\tau) + \cdots$$
$$p(\xi,\tau) = p_0(\xi,\tau) + \varepsilon p_1(\xi,\tau) + \varepsilon^2 p_2(\xi,\tau) + \cdots, \qquad (11)$$

we get various equations in increasing powers of ε. The leading order problem gives the dispersion relation $\omega = ck$, where $c = \sqrt{1/\epsilon\mu}$. At $\mathcal{O}(\varepsilon^1)$, the group velocity is given as $\omega' = kc^2/\omega$. At $\mathcal{O}(\varepsilon^2)$, we obtain

$$i\frac{\partial q_0}{\partial\tau} + \frac{\omega''}{2}\frac{\partial^2 q_0}{\partial\xi^2} + \frac{\omega c^2}{2}(\hat{\alpha}\mu|q_0|^2 + \epsilon\hat{\beta}|p_0|^2)q_0 = 0 \qquad (12)$$
$$i\frac{\partial p_0}{\partial\tau} + \frac{\omega''}{2}\frac{\partial^2 p_0}{\partial\xi^2} + \frac{\omega c^2}{2}(\epsilon\hat{\beta}|p_0|^2 + \hat{\alpha}\mu|q_0|^2)p_0 = 0, \qquad (13)$$

where $\omega'' = (c^2 - \omega'^2)/\omega$, τ is the slow time, and ξ the slow space variable moving at the linear group velocity. By rescaling τ, ξ and the amplitudes q_0, p_0 according to

$$\xi = X, \qquad T = \omega''\tau/2, \qquad (14)$$
$$Q = \sqrt{|\Lambda_q/\omega''|}q_0, \qquad P = \sqrt{|\Lambda_p/\omega''|}p_0 \qquad (15)$$

where $\Lambda_q = \omega c^2\mu\hat{a}$ and $\Lambda_p = \omega c^2\epsilon\hat{\beta}$, we get

$$iQ_T + Q_{XX} + \left(\sigma_q|Q|^2 + \sigma_p|P|^2\right)Q = 0 \qquad (16)$$
$$iP_T + P_{XX} + \left(\sigma_p|P|^2 + \sigma_q|Q|^2\right)P = 0, \qquad (17)$$

where $\sigma_{q,p} \equiv sign(\Lambda_{q,p})$. Eqs. (16-17) is a special case of the fairly general and frequently studied system of CNLS equations (Manakov model) known to be completely integrable for $\sigma_q = \sigma_p = \sigma$.[10] A number of bright and dark soliton solutions have been obtained for Eqs. (16-17) when $\sigma = \pm 1$.[14–19] There is evidence that single-soliton (single-hump) solutions are stable while multi-hump are not.[16] The sign of the products $\mu\hat{\alpha}$ and $\epsilon\hat{\beta}$ determine the type of nonlinear self-modulation (self-focusing or self-defocusing) effects which will occur. For $\sigma = \pm 1$ both fields experience the same type of nonlinearity.

For ϵ, $\mu > 0$ and $\hat{a}, \hat{\beta} > 0$ we have $\sigma = +1$, and the system of Eqs. (16-17) accepts solutions of the form[15]

$$Q(X,T) = u(X)e^{i\nu_q^2 T} \qquad P(X,T) = v(X)e^{i\nu_p^2 T}, \qquad (18)$$

where u, v are real functions and ν_q, u_p are real positive wave parameters. The latter is necessary, if we are interested in solitary waves that exponentially decay as $|X| \to \infty$. Introducing Eq. (18) into Eqs. (16-17) we get

$$u_{XX} - \nu_q^2 u + (u^2 + v^2)u = 0 \qquad (19)$$
$$v_{XX} - \nu_p^2 v + (v^2 + u^2)v = 0. \qquad (20)$$

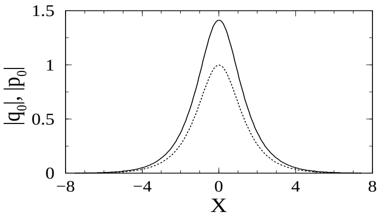

FIG. 2. Envelopes q_0 and p_0 of the compound bright soliton ($\sigma_p = \sigma_q = +1$) for $\nu = 1$ and $r = \Lambda_q/\Lambda_p = 2$ in arbitrary units, with the maximum amplitude of p_0 normalized to 1.

For $\nu_{q,p} = \nu$, Eqs. (19-20) have a one-parameter family of symmetric and single-humped soliton solutions (Fig. 2)[15–17]

$$u(X) = \pm v(X) = \nu\,\mathrm{sech}(\nu X). \tag{21}$$

There are also periodic solutions of the form

$$u(X) = A\cos(BX) \qquad v(X) = A\sin(BX), \tag{22}$$

where $A = \sqrt{\nu^2 + B^2}$ with B an arbitrary parameter. Now without loss of generality, we take $\nu_q = 1$ and denote ν_p as ν. This can always be acheived by a rescaling of variables u, v and X. Then, for $0 < \nu < 1$ there is another, in general asymmetric, one-parameter family of solutions for each fixed ν[15,18]

$$u(X) = \sqrt{2(1-\nu^2)}\cosh(\nu X)/\kappa \tag{23}$$

$$v(X) = -\nu\sqrt{2(1-\nu^2)}\sinh(X - X_0)/\kappa, \tag{24}$$

where

$$\kappa = \cosh(X - X_0)\cosh(\nu X) - \nu\sinh(X - X_0)\sinh(\nu X)$$

where X_0 is an arbitrary parameter. For $X_0 = 0$, u becomes symmetric and v antisymmetric.

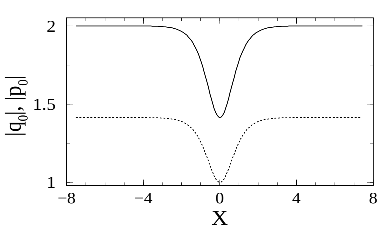

FIG. 3. Envelopes q_0 and p_0 of the compound dark soliton ($\sigma_p = \sigma_q = -1$) for $k = 1$ and $r = \Lambda_q/\Lambda_p = 2$ in arbitrary units, with the minimum amplitude of p_0 normalized to 1.

For $\epsilon, \mu < 0$ and $\hat{a}, \hat{\beta} > 0$ we have $\sigma = -1$, and Eqs.(16-17) accept dark soliton solutions of the form[19]

$$Q(X,T) = P(X,T) = k\,[\tanh(kX) - i]\,e^{i(kX - 5k^2 T)}, \tag{25}$$

which are localized dips on a finite-amplitude background wave, as shown in Fig. 3. In this very interesting case of LHM the electric and magnetic fields are coupled together forming a dark compound soliton. Note that the relative amplitudes are controlled by the corresponding nonlinearities and frequency. For $\sigma = -1$ Eqs. (16-17) have also solutions of the form[19]

$$Q(X,T) = u(X)e^{-i\nu_q^2 T}\;P(X,T) = v(X)e^{-i\nu_p^2 T}, \tag{26}$$

where u, v are real functions and ν_q, ν_p are real positive wave parameters. Introducing Eqs. (26) into Eqs. (16-17), we get

$$u_{XX} + \nu_q^2 u - (u^2 + v^2)u = 0 \tag{27}$$

$$v_{XX} + \nu_p^2 v - (v^2 + u^2)v = 0. \tag{28}$$

For $\nu_{q,p} = \nu$, Eqs. (27-28) allow for kink-shaped localized soliton solutions[20]

$$u(X) = \pm v(X) = (\nu/\sqrt{2})\tanh(\nu X/\sqrt{2}), \tag{29}$$

as can be seen in Fig. 4. In the context of the propagation of two polarization components of a transverse EM wave in a Kerr-type medium, the solution of this kind is often called a polarization domain wall. There are also periodic solutions of the form

$$u(X) = A\cos(BX) \qquad v(X) = A\sin(BX), \tag{30}$$

where $A = \sqrt{\nu^2 - B^2}$ with $B < \nu$ an arbitrary parameter.

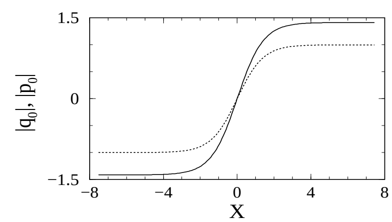

FIG. 4. Envelopes q_0 and p_0 of the kink-shaped compound soliton ($\sigma_p = \sigma_q = -1$) for $\nu = 1$ and $r = \Lambda_q/\Lambda_p = 2$ in arbitrary units, with the maximum amplitude of p_0 normalized to 1.

In order to check the validity of the Kerr-type approximation for the magnetic nonlinearity, we performed extended numerical simulations of the complete Eqs. (6 - 7)for the fields of Eqs. (8 -9) using the full expression for μ_{eff}, with ω_{0NL} obtained analytically from Eq. (5), and slightly different normalization. In the simulations we used the exact solitons of Eq. (21) for right-handed medium (RHM) and Eq. (25) for LHM as initial conditions and tested their subsequent time evolution and resulting stability. The Q-field amplitudes are shown in Fig. 5 for a right- (two right figures) and a left- (two left figures) handed medium with $\Omega = 1.2$ and 3.0, respectively. We take the (average) amplitude of Q to be unity for the right-handed medium, while the amplitude of the background wave of Q is unity for the LHM. Similar results are obtained for the field P. In the left part of Fig. 5, two bright soliton solutions of the form of Eq. (21) are shown, with their difference being the chosen value of ν

in the initial conditions. For relative high ν the soliton is still stable, but its amplitude is strongly oscillating. For relatively low ν the soliton is propagating practically undisturbed. In the right part of Fig. 5 two dark soliton solutions of the form of Eq. (25) are shown, with their difference being the chosen value of the constant k in the initial conditions. Again, for relatively high k the soliton developes strong osicllations and deforms as time progresses while for relatively low k the soliton is propagating practically undisturbed. The numerics thus demonstrates clearly that the analytical solutions of Eqs. (12-13) are good approximate solutions for the complete problem at relatively small fields while at larger amplitudes the exact solitons deform. This is expected since magnetic nonlinearity ceases to be of Kerr-type and saturation effects become more important.

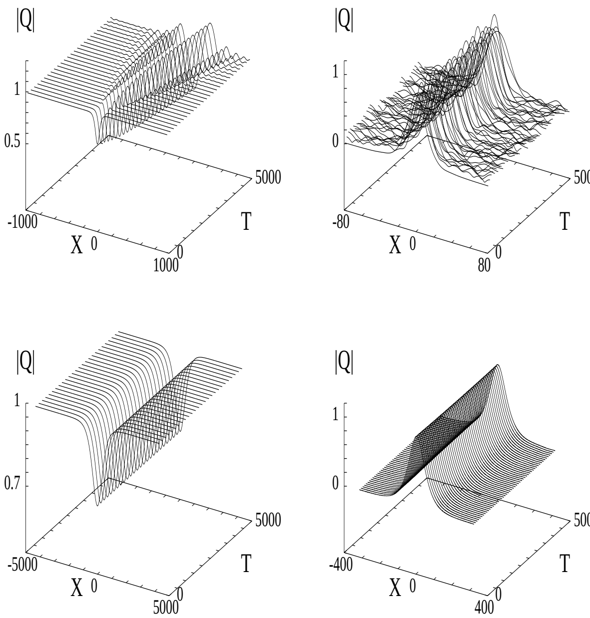

FIG. 5. Space-time evolution of solitons. Right: Bright solitons (RHM) with high (upper) and low (lower) amplitude ν. Left: Dark solitons (LHM) with high (upper) and low (lower) amplitude k.

In conclusion, we obtained a system of CNLS equations equivalent to the Manakov model describing the propagation of EM waves in a nonlinear LHM (as well as in a nonlinear RHM) for relatively small fields. Unlike a recent article,[21] where only magnetic nonlinearity was considered in the propagation of EM waves in LHMs, in the present work both magnetic and dielectric nonlinearities were retained. The present analysis however does not address the issue of the switching effect between normal medium (RHM) and LHM properties. Although not obvious from the Eqs. (3 - 5), the reduction to the Manakov form becomes possible after a proper approximation of the complex Eqs. (4) and (5). It turns out that for the choice of parameters corresponding to LHM, the system admits compound dark soliton solutions. For the choice of parameters corresponding to normal medium, the system admits compound bright soliton solutions. Reference to compound "dark" and "bright" solitons we mean that both soliton components, i.e. both the envelopes of the electric and the magnetic fields, are either both dark or both bright. The described effects are not limited to the specific SRR - wire system. They should also be present in other nonlinear LHM designs, such as photonic crystals,[22] coupled nanowire systems,[23] the transmission line systems,[24] and photonic systems.[25] The case where the fields in the medium experience different type of nonlinearity, leading to $\sigma_q = -\sigma_p = 1$ or $\sigma_p = -\sigma_q = 1$ corresponds to a medium of positive ϵ_{eff} and negative μ_{eff} or negative ϵ_{eff} and positive μ_{eff}, respectively. This interesting case where the Manakov system does not seem to be integrable will be treated numerically in a following publication.

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